

HOMOTOPY AND COMMUTATIVITY PRINCIPLE

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1. Introduction

R will denote a commutative ring with $1 \neq 0$ in this article, unless stated otherwise.

The subject of injective stability for the linear group (i.e. $K_1(R)$) began in the famous paper of Bass–Milnor–Serre ([8]) where it was shown, in essence, that large sized stably elementary matrices were actually elementary matrices. This was shown by showing that the sequence (of pointed sets)

$$\cdots \longrightarrow \frac{SL_n(R)}{E_n(R)} \longrightarrow \frac{SL_{n+1}(R)}{E_{n+1}(R)} \longrightarrow \cdots$$

stabilizes. The estimate they got was $n = 3$, when $\dim(R) = 1$, and for $n \geq \max\{3, d + 3\}$ otherwise. They conjectured that the correct bound for the linear quotients should be $n \geq \max\{3, d + 2\}$; which was established by L.N. Vaserstein in [38].

In [35] A.A. Suslin established the normality of the elementary linear subgroup $E_n(R)$ in $GL_n(R)$, for $n \geq 3$. This was a major surprise at that time as it was known due to the work of P.M. Cohn in [12] that in general $E_2(R)$ is not normal in $GL_2(R)$. This is the initial precursor to study the non-stable K_1 groups $\frac{SL_n(R)}{E_n(R)}$, $n \geq 3$.

This theorem can also be got as a consequence of the Local-Global Principle of D. Quillen (for projective modules) in [26]; and its analogue for the linear group of elementary matrices $E_n(R[X])$, when $n \geq 3$ due to A. Suslin in [35]. In fact, in [9] it is shown that, in some sense, the normality property of the elementary group $E_n(R)$ in $SL_n(R)$ is equivalent to having a Local-Global Principle for $E_n(R[X])$.

In [6], A. Bak proved the following beautiful result:

Theorem 1.1. (A. Bak) *For an almost commutative ring R with identity with centre $C(R)$. The group $\frac{SL_n(R)}{E_n(R)}$ is nilpotent of class atmost $\delta(C(R)) + 3 - n$, where $\delta(C(R)) < \infty$ and $n \geq 3$, where $\delta(C(R))$ is the Bass–Serre-dimension of $C(R)$.*

This theorem, which is proved by a localisation and completion technique, which evolved from an adaptation of the proof of the Suslin’s K_1 -analogue of Quillen’s Local-Global Principle, was the starting point of our investigation. In this paper, we show (see Corollary 2.20)

Theorem 1.2. *Let R be a local ring, and let $A = R[X]$. Then the group $\frac{SL_n(A)}{E_n(A)}$ is an abelian group for $n \geq 3$.*

This theorem is a simple consequence of the following principle (see Theorem 2.19):

Theorem 1.3. (*Homotopy and commutativity principle*) : Let R be a commutative ring. Let $\alpha \in SL_n(R)$, $n \geq 3$, be homotopic to the identity. Then, for any $\beta \in SL_n(R)$, $\alpha\beta = \beta\alpha\varepsilon$, for some $\varepsilon \in E_n(R)$.

This principle is a consequence of the Quillen–Suslin Local-Global principle; and using a *non-symmetric* application of it as done by A. Bak in [6].

The existence of a Local-Global Principle enables us to prove similar results in various groups.

We restrict ourselves to the classical symplectic, orthogonal groups (and their relative versions); and to the automorphism groups of a projective module (with a unimodular element), a symplectic module (with a hyperbolic summand), and an orthogonal module (with a hyperbolic symmand).

However, our results can be extended to other Chevalley groups, relative Chevalley groups, reductive groups, etc. where such Local-Global Principles exist due to results of E. Abe in [1], A. Stepanov in [4], [34], Asok–Hoyois–Wendt in [5], A. Stavrova in [33], respectively.

We could show that the symplectic quotients were abelian, but we could only establish that the orthogonal quotients are solvable of length at most two. We do believe that the orthogonal quotient groups are also abelian; and prove this when the base ring is a regular local ring containing a field.

In ([20], Theorem 4.1), W. van der Kallen has described an abelian group structure on the orbit space of unimodular rows under elementary action $\frac{Um_n(R)}{E_n(R)}$, when $n \geq 3$ and $d \leq 2n - 4$, where d is the dimension of R . In the paper [19] he does it in the case when $n = d + 1$, where d is the dimension of R ; thereby extending the seminal work of L.N. Vaserstein in ([39], Theorem 5.2), when $d = 2$. His estimates come from similar estimates being true in case when R is the ring of continuous real valued functions on a compact space X .

Let $Comp_r(R)$ denote the subset of $Um_r(R)$ consisting of the (completable) unimodular rows which can be completed to a matrix of determinant one. One of the interesting application of Theorem 1.2 is that the orbit set of completable unimodular rows over $R[X]$, when R is a local ring, modulo the elementary action has an abelian group structure under matrix multiplication. (See Theorem 2.33.)

In particular, if one believes that the Bass–Suslin conjecture that unimodular rows over a polynomial extension of a local ring is true, then one would have an abelian group structure on the orbit space $\frac{Um_n(R[X])}{E_n(R[X])}$. The only restriction on size is $n \geq 3$. Since one does know the truth of the Bass–Suslin conjecture when dimension R is 3 and $2R = R$ (see [28], [29]); one does get $\frac{Um_3(R[X])}{E_3(R[X])}$ has an abelian group structure, when R is a local ring of dimension 3 in which 2 is invertible.

Is there a (perhaps \mathbb{A}^1 –homotopy) interpretation of this result from a topological point of view?

2. Linear and Symplectic group

First we collect some definitions and some known results, and set notations which will be used throughout the paper.

Definition 2.1. Special linear group $SL_n(R)$: The subgroup of the General linear group $GL_n(R)$, of $n \times n$ invertible matrices of determinant 1.

Definition 2.2. Elementary group $E_n(R)$: The subgroup of all matrices of $GL_n(R)$ generated by $\{e_{ij}(\lambda) : \lambda \in R, \text{ for } i \neq j\}$, where $e_{ij}(\lambda) = I_n + \lambda E_{ij}$ and E_{ij} is the matrix with 1 on the ij^{th} place and 0's elsewhere.

Notation 2.3. Let $\psi_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\psi_n = \psi_{n-1} \perp \psi_1$; and $\phi_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\phi_n = \phi_{n-1} \perp \phi_1$, for $n > 1$.

Notation 2.4. Let σ be the permutation of the natural numbers given by $\sigma(2i) = 2i - 1$ and $\sigma(2i - 1) = 2i$.

Definition 2.5. **Symplectic group** $Sp_{2m}(R)$: the group of all $2m \times 2m$ matrices $\{\alpha \in GL_{2m}(R) \mid \alpha^t \psi_m \alpha = \psi_m\}$.

Definition 2.6. **Elementary Symplectic group** $ESp_{2m}(R)$: We define for $1 \leq i \neq j \leq 2m$, $z \in R$,

$$se_{ij}(z) = \begin{cases} I_{2m} + zE_{ij}, & \text{if } i = \sigma(j); \\ I_{2m} + zE_{ij} - (-1)^{i+j} zE_{\sigma(j)\sigma(i)}, & \text{if } i \neq \sigma(j). \end{cases}$$

It is easy to verify that all these matrices belong to $Sp_{2m}(R)$. We call them the elementary symplectic matrices over R . The subgroup generated by them is called the elementary symplectic group and is denoted by $ESp_{2m}(R)$.

Definition 2.7. **Orthogonal group** $O_{2m}(R)$: the group of all $2m \times 2m$ matrices $\{\alpha \in GL_{2m}(R) \mid \alpha^t \phi_m \alpha = \phi_m\}$.

Definition 2.8. **Elementary Orthogonal group** $EO_{2m}(R)$: We define for $1 \leq i \neq j \leq 2m$, $z \in R$,

$$o_{ij}(z) = I_{2m} + zE_{ij} - zE_{\sigma(j)\sigma(i)}, \text{ if } i \neq \sigma(j).$$

It is easy to verify that all these matrices belong to $O_{2m}(R)$. We call them the elementary orthogonal matrices over R . The subgroup generated by them is called the elementary orthogonal group and is denoted by $EO_{2m}(R)$.

Notation 2.9. Let R be a commutative ring with identity. In this paper $M(n, R)$ will denote the set of all $n \times n$ matrices over R , $G(n, R)$ will denote either the linear group $GL_n(R)$ or the symplectic group $Sp_{2m}(R)$, where $2m = n$. $E(n, R)$ will denote either elementary subgroups $E_n(R)$ or elementary symplectic subgroup $ESp_{2m}(R)$. And, $S(n, R)$ will denote either the special linear group $SL_n(R)$ or the symplectic group $Sp_{2m}(R)$.

Convention 2.10. Throughout this paper, we will assume size of the matrix is $n \geq 3$ in the linear case, $n \geq 4$ in symplectic and $n \geq 6$ in orthogonal case, unless stated otherwise.

2.11

Lemma 2.11. (L.N. Vaserstein) ([39, Lemma 5.5]) For an associative ring R with identity, and for any natural number m

$$E_{2m}(R)e_1 = (Sp_{2m}(R) \cap E_{2m}(R))e_1.$$

2.12

Remark 2.12. It was observed in ([11, Lemma 2.13]) that Vaserstein's proof actually shows that $E_{2m}(R)e_1 = ESp_{2m}(R)e_1$.

In view of above remark, or otherwise, one has:

2.13

Lemma 2.13. ([24, Chapter 1, Proposition 5.4]) *Let $c = (c_1, \dots, c_n)$ be a unimodular row over a semilocal ring R . Then $(c_1, \dots, c_n) \in e_1 E(n, R)$; for $n \geq 2$, i.e.*

$$(c_1, \dots, c_n) \stackrel{E(n, R)}{\sim} (1, 0, \dots, 0) \text{ for } n \geq 2.$$

The next Lemma is well-known. We include it with a proof, for completeness.

2.14

Lemma 2.14. (Only for the linear and the symplectic group) *Let R be a local ring. For $n \geq 2$, $S(n, R) = E(n, R)$, where $n = 2m$, m is any natural number.*

Proof : For the linear case we prove the result by induction on n . When $n = 2$, it is obvious as $SL_2(R) = E_2(R)$. For $n > 2$, let $\alpha \in SL_n(R)$, By Lemma 2.13,

$$\alpha \stackrel{E_n(R)}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & \alpha' \end{bmatrix}, \alpha' \in E_n(R).$$

By induction hypothesis we have $\alpha' \in E_{n-1}(R)$, thus $\alpha \in E_n(R)$.

In the symplectic case let $\tau \in Sp_{2m}(R)$. We use the induction on m , for $m = 1$, $SL_2(R) = E_2(R) = Sp_2(R) = ESp_2(R)$. Since $\tau e_1 \in E_{2m}(R)e_1$, by Lemma 2.11 (and remark following it), $\tau e_1 \in ESp_{2m}(R)e_1$. Let $\tau e_1 = \varepsilon_1 e_1$ for some $\varepsilon_1 \in ESp_{2m}(R)$. Hence $\varepsilon_1^{-1} \tau e_1 = e_1$. Therefore, we can find $\varepsilon_2 \in ESp_{2m}(R)$ such that $\varepsilon_2^{-1} \varepsilon_1^{-1} \tau = I_2 \perp \tau^*$ for some $\tau^* \in Sp_{2m-2}(R)$. By induction $\tau^* \in ESp_{2m-2}(R)$. Repeating this process we can reduce τ to a 2×2 symplectic matrix. □

We begin with some initial observations:

2.15

Lemma 2.15. *Let R be a local ring and $\alpha(X), \beta(X) \in S(n, R[X])$. Then the commutator,*

$$[\alpha(X), \beta(X)] \in [\alpha(X)\alpha(0)^{-1}, \beta(X)\beta(0)^{-1}]E(n, R[X])$$

Proof : Since R is a local ring, $S(n, R) = E(n, R)$ for all $n \geq 2$, by Lemma 2.14. Thus $\alpha(0), \beta(0) \in E(n, R)$. Let $s = \alpha(X)\alpha(0)^{-1}$, $t = \beta(X)\beta(0)^{-1}$. Then,

$$\begin{aligned} [\alpha(X), \beta(X)] &= [\alpha(X)\alpha(0)^{-1}\alpha(0), \beta(X)\beta(0)^{-1}\beta(0)] \\ &= s\alpha(0)t\beta(0)(s\alpha(0))^{-1}(t\beta(0))^{-1} \\ &= sts^{-1}t^{-1}(tst^{-1}\alpha(0)ts^{-1}t^{-1})(ts\beta(0)\alpha(0)^{-1}s^{-1}t^{-1})(t\beta(0)^{-1}t^{-1}). \end{aligned}$$

Since $E(n, R[X])$ is a normal subgroup of $S(n, R[X])$, hence $(tst^{-1}\alpha(0)ts^{-1}t^{-1}), (ts\beta(0)\alpha(0)^{-1}s^{-1}t^{-1}), (t\beta(0)^{-1}t^{-1}) \in E(n, R[X])$. □

2.16

Theorem 2.16. (Local-Global Principle for the Linear groups) ([35, Theorem 3.1]) *Let R be a commutative ring, $n \geq 3$ and $\alpha \in GL_n(R[X])$ such that $\alpha(0) = Id$. Then α lies in $E_n(R[X])$ if and only if for every maximal ideal \mathfrak{m} of R , the canonical image of α in $GL_n(R_{\mathfrak{m}}[X])$ lies in $E_n(R_{\mathfrak{m}}[X])$.*

2.17

Theorem 2.17. (Local-Global Principle for the Symplectic groups) ([23, Theorem 3.6]) *Let $m \geq 2$ and $\alpha(X) \in Sp_{2m}(R[X])$, with $\alpha(0) = Id$. Then $\alpha(X) \in ESp_{2m}(R[X])$ if and only if for any maximal ideal $\mathfrak{m} \subset R$, the canonical image of $\alpha(X) \in Sp_{2m}(R_{\mathfrak{m}}[X])$ lies in $ESp_{2m}(R_{\mathfrak{m}}[X])$.*

For a uniform proof of above Theorems see ([9])

Definition 2.18. Let R be a ring. A matrix $\alpha \in S(n, R)$ is said to be homotopic to identity if there exists a matrix $\gamma(X) \in S(n, R[X])$ such that $\gamma(0) = Id$ and $\gamma(1) = \alpha$.

2.19

Theorem 2.19. Let $\alpha \in S(n, R)$ be homotopic to identity. Then $[\alpha, \beta] \in E(n, R)$, $\forall \beta \in S(n, R)$.

Proof : Since α is homotopic to identity, there exists $\gamma \in S(n, R[X])$ such that $\gamma(0) = Id$, $\gamma(1) = \alpha$. Define,

$$\delta(X) = [\gamma(X), \beta].$$

Note that $\delta(0) = Id$, and for every maximal ideal \mathfrak{m} of R ,

$$\delta(X)_{\mathfrak{m}} = [\gamma(X)_{\mathfrak{m}}, \beta_{\mathfrak{m}}].$$

By Lemma 2.14, $\beta_{\mathfrak{m}} \in E(n, R_{\mathfrak{m}})$ and since $E(n, R)$ is normal in $S(n, R)$, we have $\delta(X)_{\mathfrak{m}} \in E(n, R_{\mathfrak{m}}[X])$. Thus by Theorem 2.16 (respectively Theorem 2.17), $\delta(X) = [\gamma(X), \beta] \in E(n, R[X])$, which implies

$$\delta(1) = [\gamma(1), \beta] = [\alpha, \beta] \in E(n, R).$$

□

2.20

Corollary 2.20. Let R be a local ring. Then the group $\frac{S(n, R[X])}{E(n, R[X])}$ is an abelian group.

Proof : Let $\alpha(X), \beta(X) \in S(n, R[X])$, we need to prove that $[\alpha(X), \beta(X)] \in E(n, R[X])$. In view of Lemma 2.15, we may assume that $\alpha(0) = \beta(0) = Id$.

Define, $\gamma(X, T) = \alpha(XT)$. Clearly $\gamma(X, 0) = Id$ and $\gamma(X, 1) = \alpha(X)$; thus $\alpha(X)$ is homotopic to identity. Thus, one gets the desired result by Theorem 2.19.

□

The Relative case. Let I be an ideal of a ring R , we shall denote by $GL_n(R, I)$ the kernel of the canonical mapping $GL_n(R) \longrightarrow GL_n\left(\frac{R}{I}\right)$. Let $SL_n(R, I)$ denotes the subgroup of $GL_n(R, I)$ of elements of determinant 1.

Definition 2.21. The Relative Groups $E_n(I)$, $E_n(R, I)$: Let I be an ideal of R . The elementary group $E_n(I)$ is the subgroup of $E_n(R)$ generated as a group by the elements $e_{ij}(x)$, $x \in I$, $1 \leq i \neq j \leq n$. The relative elementary group $E_n(R, I)$ is the normal closure of $E_n(I)$ in $E_n(R)$.

Definition 2.22. The Relative Groups $ESp_{2m}(I)$, $ESp_{2m}(R, I)$: Let I be an ideal of R . The elementary symplectic group $ESp_{2m}(I)$ is the subgroup of $ESp_{2m}(R)$ generated as a group by the elements $se_{ij}(x)$, $x \in I$, $1 \leq i \neq j \leq 2m$.

The relative elementary symplectic group $ESp_{2m}(R, I)$ is the normal closure of $ESp_{2m}(I)$ in $ESp_{2m}(R)$.

Definition 2.23. The Relative Groups $EO_{2m}(I)$, $EO_{2m}(R, I)$: Let I be an ideal of R . The elementary orthogonal group $EO_{2m}(I)$ is the subgroup of $EO_{2m}(R)$ generated as a group by the elements $oe_{ij}(x)$, $x \in I$, $1 \leq i \neq j \leq 2m$.

The relative elementary orthogonal group $EO_{2m}(R, I)$ is the normal closure of $EO_{2m}(I)$ in $EO_{2m}(R)$.

Notation 2.24. Let R be a commutative ring with identity and I be an ideal of R . In this paper $E(n, R, I)$ will denote either relative elementary group $E_n(R, I)$ or relative elementary symplectic group $ESp_{2m}(R, I)$, and $S(n, R, I)$ will denote either $SL_n(R, I)$ or the relative symplectic group $Sp_{2m}(R, I)$ where $2m = n$.

Definition 2.25. Excision Ring : Let R be a ring and I be an ideal of R . The excision ring $R \oplus I$, has coordinate wise addition and multiplication is given as follows:

$$(r, i) \cdot (s, j) = (rs, rj + si + ij), \text{ where } r, s \in R \text{ and } i, j \in I.$$

The multiplicative identity of this group is $(1, 0)$ and the additive identity is $(0, 0)$.

2.29

Lemma 2.26. (Anjan Gupta) (see [15, Lemma 4.3]) Let (R, \mathfrak{m}) be a local ring. Then the excision ring $R \oplus I$ with respect to a proper ideal $I \subsetneq R$ is also a local ring with maximal ideal $\mathfrak{m} \oplus I$.

2.30

Lemma 2.27. Let R be a local ring and I be a proper ideal of R (i.e. $I \neq R$). Then $S(n, R, I) = E(n, R, I)$ for all $n \geq 1$.

Proof : Let $\sigma \in S(n, R, I)$, we can write $\sigma = Id + \sigma'$, for some $\sigma' \in M_n(I)$. Let $\tilde{\sigma} = (Id, \sigma') \in S(n, R \oplus I, 0 \oplus I)$. By Anjan's Lemma, $R \oplus I$ is a local ring, thus by Lemma 2.14,

$$\tilde{\sigma} \in E(n, R \oplus I) \cap S(n, R \oplus I, 0 \oplus I) = E(n, R \oplus I, 0 \oplus I)$$

as $\frac{R \oplus I}{0 \oplus I} \simeq R$ is a retract of $R \oplus I$. Thus,

$$\tilde{\sigma} = \prod_{k=1}^m \beta_k g e_{i_k j_k} (0, a_k) \beta_k^{-1}, \beta_k \in E(n, R \oplus I), a_k \in I.$$

Now, consider the homomorphism

$$\begin{aligned} f : R \oplus I &\longrightarrow R \\ (r, i) &\longmapsto r + i. \end{aligned}$$

This f induces a map

$$\tilde{f} : E(n, R \oplus I, 0 \oplus I) \longrightarrow E(n, R)$$

Clearly,

$$\begin{aligned} \sigma &= \tilde{f}(\tilde{\sigma}) \\ &= \prod_{k=1}^m \gamma_k g e_{i_k j_k} (0 + a_k) \gamma_k^{-1} \\ &= \prod_{k=1}^m \gamma_k g e_{i_k j_k} (a_k) \gamma_k^{-1} \in E(n, R, I); \text{ since } a_k \in I, \end{aligned}$$

where, $\gamma_k = \tilde{f}(\beta_k)$

□

rel

Theorem 2.28. *Let R be a ring and I be a proper ideal (i.e. $I \neq R$) of R . Let $\alpha \in S(n, R, I)$ which is homotopic to identity relative to an extended ideal. Then $[\alpha, \beta] \in E(n, R, I), \forall \beta \in S(n, R, I)$.*

Proof : Let $\alpha, \beta \in S(n, R, I)$ be such that α is homotopic to identity relative to an extended ideal. We can write $\alpha = Id + \alpha', \beta = Id + \beta'$ for some $\alpha', \beta' \in M_n(I)$. Let $\sigma = [\alpha, \beta] = Id + \sigma'$ for some $\sigma' \in M_n(I)$. Let $\tilde{\sigma} = (Id, \sigma') \in S(n, R \oplus I, 0 \oplus I)$. In view of Theorem 2.19,

$$\tilde{\sigma} \in E(n, R \oplus I) \cap S(n, R \oplus I, 0 \oplus I) = E(n, R \oplus I, 0 \oplus I)$$

as $\frac{R \oplus I}{0 \oplus I} \simeq R$ is a retract of $R \oplus I$. Thus,

$$\tilde{\sigma} = \prod_{k=1}^m \varepsilon_k g e_{i_k j_k} (0, a_k) \varepsilon_k^{-1}, \quad \varepsilon_k \in E(n, R \oplus I), \quad a_k \in I.$$

Now, consider the homomorphism

$$\begin{aligned} f : R \oplus I &\longrightarrow R \\ (r, i) &\longmapsto r + i. \end{aligned}$$

This f induces a map

$$\tilde{f} : E(n, R \oplus I, 0 \oplus I) \longrightarrow E(n, R)$$

Clearly,

$$\begin{aligned} \sigma &= \tilde{f}(\tilde{\sigma}) \\ &= \prod_{k=1}^m \gamma_k g e_{i_k j_k} (0 + a_k) \gamma_k^{-1} \\ &= \prod_{k=1}^m \gamma_k g e_{i_k j_k} (a_k) \gamma_k^{-1} \in E(n, R, I); \text{ since } a_k \in I, \end{aligned}$$

where, $\gamma_k = \tilde{f}(\varepsilon_k)$

□

In view of the well-known Swan–Weibel homotopy trick ([24, Appendix 3]), one has:

grad

Corollary 2.29. *Let $A = \bigoplus_{d \geq 0} A_d$ be a graded ring with augmentation ideal $A_+ = \bigoplus_{d \geq 1} A_d$. Then $\frac{S(n, A, A_+)}{E(n, A, A_+)}$ is an abelian group.*

Proof : Consider the ring homomorphism

$$\varphi : A \longrightarrow A[T]$$

$$a_0 + a_1 + \cdots \longmapsto a_0 + a_1 T + \cdots$$

Note that φ is an injective ring homomorphism. For any element $\alpha = (\alpha_{ij}) \in S(n, A, A_+)$, define $\alpha(T) = (\varphi(\alpha_{ij}))$. Now, note that $\alpha(0) = Id$ and $\alpha(1) = \alpha$. Thus by Theorem 2.28 $\frac{S(n, A, A_+)}{E(n, A, A_+)}$ is an abelian group.

□

Corollary 2.30. *Let A be an affine algebra of dimension $d \geq 2$ over a perfect C_1 field k and $(d+1)!$ is a unit in k . Let $\sigma \in SL_{d+1}(A)$ be a stably elementary matrix. Then $[\sigma, \tau] \in E_{d+1}(A)$, for all $\tau \in SL_{d+1}(A)$.*

Proof : In view of ([30, Theorem 3.4]), there exists a matrix $\sigma(X) \in SL_{d+1}(A[X])$ with $\sigma(0) = Id$ and $\sigma(1) = \sigma$. Now, we are through by Theorem 2.19. \square

Corollary 2.31. *Let A be an affine algebra of dimension $d \geq 3$ over an algebraically closed field k and $d!$ is a unit in k . Then the group $\frac{SL_d(A) \cap E_{d+1}(A)}{E_d(A)}$ is an abelian group.*

Proof : Let $\sigma \in SL_d(A) \cap E_{d+1}(A)$. In view of ([14, Corollary 7.7]), σ is homotopic to identity. Thus we are through by Theorem 2.19. \square

Corollary 2.32. *Let A be an affine algebra of even dimension d over a field k of cohomological dimension ≤ 1 . If $(d+1)!A = A$ and $4|d$, then $\frac{Sp_d(A) \cap ESp_{d+2}(A)}{ESp_d(A)}$ is an abelian group.*

Proof : Let $\sigma \in Sp_d(A) \cap ESp_{d+2}(A)$. In view of ([10, Theorem 1]), σ is symplectic homotopic to Id . Thus we are through by Theorem 2.19. \square

The reader should contrast the next result with the results ([30, Theorem 5.1]) and ([20, Proposition 7.10]) of W. van der Kallen.

2.25

Theorem 2.33. *Let $A = \bigoplus_{d \geq 0} A_d$ be a graded ring with augmentation ideal $A_+ = \bigoplus_{d \geq 1} A_d$. Then for $n \geq 3$, $\frac{Comp_n(A, A_+)}{E_n(A, A_+)}$ has an abelian group structure under matrix multiplication. In particular, for $n \geq 3$, the first row map*

$$\begin{aligned} SL_n(A, A_+) &\longrightarrow \frac{Comp_n(A, A_+)}{E_n(A, A_+)} \\ \sigma &\longmapsto [e_1 \sigma] \end{aligned}$$

is a group homomorphism.

Proof : Since $\frac{SL_n(A, A_+)}{E_n(A, A_+)}$ is an abelian group, it is enough to prove that matrix multiplication gives a well defined (abelian) operation on $\frac{Comp_n(A, A_+)}{E_n(A, A_+)}$. Let $v, u \in \frac{Comp_n(A, A_+)}{E_n(A, A_+)}$ such that

$$\begin{aligned} v &= e_1 \alpha = e_1 \alpha' ; \alpha, \alpha' \in SL_n(A, A_+) \\ u &= e_1 \beta = e_1 \beta' ; \beta, \beta' \in SL_n(A, A_+) \end{aligned}$$

To get a well-defined multiplication on $\frac{Comp_n(A, A_+)}{E_n(A, A_+)}$, we need to prove that $[e_1 \alpha \beta] = [e_1 \alpha' \beta']$. By Corollary 2.29,

$$[e_1 \alpha \beta] = [e_1 \alpha' \beta] = [e_1 \beta \alpha'] = [e_1 \beta' \alpha'] = [e_1 \alpha' \beta'].$$

\square

Corollary 2.34. *Let A be a commutative ring and $Comp_n(A)$ denote the subset of $Um_n(A)$ consisting of those unimodular rows which can be completed to an invertible matrix of determinant 1. If R is a local ring, then $\frac{Comp_n(R[X])}{E_n(R[X])}$ has an abelian group structure, under matrix multiplication for $n \geq 3$.*

Remark 2.35. *There exist examples which show that $\frac{Comp_n(A, A_+)}{E_n(A, A_+)}$ is non-trivial, for some graded ring A :*

Let $R = k[X, Y, Z]/(Z^7 - X^2 - Y^3)$, where k is \mathbb{C} or any sufficiently large field of characteristic $\neq 2$. It is shown in ([31], page 4) that if $B = R[T, T^{-1}]$, then there is a maximal ideal \mathfrak{m} for which $NW_E(B_{\mathfrak{m}}) \neq 0$. By ([37], Theorem 5.2 (b)) the Vaserstein symbol $\frac{Um_3(B_{\mathfrak{m}}[W])}{E_3(B_{\mathfrak{m}}[W])} \rightarrow W_E(B_{\mathfrak{m}}[W])$ is onto. Hence, there exists a unimodular row $v(W) \in Um_3(B_{\mathfrak{m}}[W])$ which is not elementarily completable. However, by [29], $v(W)$ is completable.

2.31

Theorem 2.36. (Local Global Principle for Extended Ideals) ([3, Theorem 1.3]) *Let $\alpha(X) \in G(n, R[X], I[X])$ be such that $\alpha(0) = Id$. If $\alpha_{\mathfrak{m}}(X) \in E(n, R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X])$ for every maximal \mathfrak{m} of R , then $\alpha(X) \in E(n, R[X], I[X])$.*

2.32

Lemma 2.37. *Let R be a local ring and I be a proper ideal of R (i.e. $I \neq R$). Then the group $\frac{S(n, R[X], I[X])}{E(n, R[X], I[X])}$ is an abelian group.*

Proof : Let $\alpha(X), \beta(X) \in S(n, R[X], I[X])$, we need to prove that $[\alpha(X), \beta(X)] \in E(n, R[X], I[X])$. In view of Lemma 2.15, we may assume that $\alpha(0) = \beta(0) = Id$. Define,

$$\gamma(X, T) = [\alpha(XT), \beta(X)]$$

Note that, $\gamma(X, 0) = Id$ and for every maximal ideal \mathfrak{m} of $R[X]$, $\gamma(X, T)_{\mathfrak{m}} = [\alpha(XT)_{\mathfrak{m}}, \beta(X)_{\mathfrak{m}}]$, by Lemma 2.27, $\beta(X)_{\mathfrak{m}} \in E(n, R[X]_{\mathfrak{m}}, I[X]_{\mathfrak{m}}) \subseteq E(n, R[X]_{\mathfrak{m}}, I[X]_{\mathfrak{m}}[T])$; and since E_n is normal in SL_n , we have,

$$\gamma(X, T)_{\mathfrak{m}} \in E(n, R[X]_{\mathfrak{m}}, I[X]_{\mathfrak{m}}[T]).$$

Thus by Theorem 2.36, $\gamma(X, T) \in E((n, R[X], I[X])[T])$ which implies that $\gamma(X, 1) = [\alpha(X), \beta(X)] \in E(n, R[X], I[X])$. □

Corollary 2.38. *Let A be an affine algebra of dimension d over an algebraically closed field k and $I = (a)$ be a principal ideal. Assume $(d+1)! \in k^*$, $d \equiv 1 \pmod{4}$. Then $\frac{Sp_{d-1}(A, I) \cap ESP_{d+1}(A, I)}{ESP_{d-1}(A, I)}$ is an abelian group.*

Proof : Let $\sigma \in Sp_{d-1}(A, I) \cap ESP_{d+1}(A, I)$. In view of ([16, Theorem 5.4]), σ is symplectic homotopic to Id . Thus we are through by Theorem 2.19. □

3. Transvection groups

First, we collect some definitions and some known results and set notations which will be used in this paper.

Definition 3.1. *Let M be a finitely generated module over a ring R . An element m of M is said to be unimodular in M if $Rm \cong R$ and $M \cong Rm \oplus M'$, for some R -submodule M' of M .*

Definition 3.2. *For an element $m \in M$, one can attach an ideal, called the order ideal of m in M , viz.*

$$O_M(m) = \{f(m) \mid f \in M^* = \text{Hom}(M, R)\}$$

Definition 3.3. We define a transvection of a finitely generated R -module as follows: Let M be a finitely generated R -module. Let $q \in M$ and $f \in M^*$ with $f(q) = 0$. An automorphism of M of the form $1 + f_q$ (defined by $f_q(p) = f(p)q$, for $p \in M$), will be called a **transvection** of M if either $q \in \text{Um}(M)$ or $f \in \text{Um}(M^*)$. We denote by $\text{Trans}(M)$ the subgroup of $\text{Aut}(M)$ generated by transvections of M .

Definition 3.4. Let M be a finitely generated R -module. The automorphisms of the form $(p, a) \mapsto (p + ax, a)$ and $(p, a) \mapsto (p, a + f(p))$, where $x \in M$ and $f \in M^*$, are called **elementary transvections** of $M \oplus R$. (Note that we can regard f as an element of $(M \oplus R)^*$ by defining $f(0, 1) = 0$.) By taking $q = (x, 0)$ and $f \in (M \oplus R)^*$ such that $f : (y, t) \mapsto t$ for $(y, t) \in (M \oplus R)$, one can verify that the automorphism $(p, a) \mapsto (p + ax, a)$ is in $\text{Trans}(M \oplus R)$. Similarly, by taking $q = (0, 1)$ and $f \in (M \oplus R)^*$ such that $f : (0, 1) \mapsto 0$ one can verify that the automorphism $(p, a) \mapsto (p, a + f(p))$ is in $\text{Trans}(M \oplus R)$.

The subgroup of $\text{Trans}(M \oplus R)$ generated by the elementary transvections is denoted by $\text{ETrans}(M \oplus R)$.

Definition 3.5. A **symplectic (respectively orthogonal)** R -module is a pair (P, \langle, \rangle) , where P is a projective R -module of even rank and $\langle, \rangle : P \times P \rightarrow R$ is a non-degenerate alternating (respectively symmetric) bilinear form.

Definition 3.6. Let $(P_1, \langle, \rangle_1)$ and $(P_2, \langle, \rangle_2)$ be two symplectic (respectively orthogonal) R -modules. Their **orthogonal sum** is a pair (P, \langle, \rangle) , where $P = P_1 \oplus P_2$ and the inner product is defined by $\langle (p_1, p_2), (q_1, q_2) \rangle = \langle p_1, q_1 \rangle_1 + \langle p_2, q_2 \rangle_2$. Since this form is also non-singular we shall henceforth denote (P, \langle, \rangle) by $P_1 \perp P_2$ called the orthogonal sum of $(P_1, \langle, \rangle_1)$ and $(P_2, \langle, \rangle_2)$.

Definition 3.7. For a projective R -module P of rank n , we define $\mathbb{H}(P)$ of rank $2n$ supported by $P \oplus P^*$, with form $\langle (p, f), (p', f') \rangle = f(p') - f'(p)$ for the symplectic modules and $f(p') + f'(p)$ for the orthogonal modules.

Definition 3.8. An **isometry** of a symplectic (respectively orthogonal) module (P, \langle, \rangle) is an automorphism of P which fixes the bilinear form. The group of isometries of (P, \langle, \rangle) is denoted by $\text{Sp}(P)$ for the symplectic modules and $\text{O}(P)$ for the orthogonal modules.

Definition 3.9. We define a symplectic transvection as follows: Let $\Psi : P \rightarrow P^*$ be an induced isomorphism. Let $\alpha : R \rightarrow P$ be a R -linear map defined by $\alpha(1) = u$. Then $\alpha^* \Psi$ defined by $\alpha^* \Psi(p) = \langle u, p \rangle$ is in P^* . Let $v \in P$ be such that $\alpha^* \Psi(v) = \langle u, v \rangle = 0$. An automorphism $\sigma_{(u,v)}$ of (P, \langle, \rangle) of the form

$$\sigma_{(u,v)}(p) = p + \langle u, p \rangle v + \langle v, p \rangle u + \langle u, v \rangle u$$

for $u, v \in P$ with $\langle u, v \rangle = 0$ will be called a **symplectic transvection** of (P, \langle, \rangle) if either $v \in \text{Um}(P)$ or $\alpha^* \Psi \in \text{Um}(P^*)$. Since $\langle \sigma_{(u,v)}(p(1)), \sigma_{(u,v)}(p(2)) \rangle = \langle p_1, p_2 \rangle$, $\sigma_{(u,v)} \in \text{Sp}(P, \langle, \rangle)$. Note that $\sigma_{(u,v)}^{-1}(p) = p - \langle u, p \rangle v - \langle v, p \rangle u - \langle u, v \rangle u$. The subgroup of $\text{Sp}(P, \langle, \rangle)$ generated by symplectic transvections is denoted by $\text{Trans}_{\text{Sp}}(P)$.

Definition 3.10. The symplectic transvections of $P \perp R^2$ of the form

$$\begin{aligned} (p, b, a) &\mapsto (p + aq, b - \langle p, q \rangle + a, a), \\ (p, b, a) &\mapsto (p + bq, b, a - b + \langle p, q \rangle), \end{aligned}$$

where $a, b \in R$ and $p, q \in P$, are called **elementary symplectic transvections**.

One can verify that above two maps belong to $\text{Trans}_{Sp}(P \perp R^2)$. The subgroup of $\text{Trans}_{Sp}(P \perp R^2)$ generated by elementary symplectic transvections is denoted by $E\text{Trans}_{Sp}(P \perp R^2)$.

In a similar manner we can find a transvection $\tau_{(u,v)}$ for an orthogonal module (P, \langle, \rangle) . For this we need to assume that $u, v \in P$ are **isotropic**, i.e. $\langle u, u \rangle = \langle v, v \rangle = 0$.

Definition 3.11. An automorphism $\tau_{(u,v)}$ of (P, \langle, \rangle) of the form

$$\tau_{(u,v)}(p) = p - \langle u, p \rangle v + \langle v, p \rangle u$$

for $u, v \in P$ with $\langle u, v \rangle = \langle u, u \rangle = \langle v, v \rangle = 0$ will be called an **isotropic orthogonal transvection** of (P, \langle, \rangle) if either $v \in U\mathfrak{m}(P)$ or $\alpha^* \Psi \in U\mathfrak{m}(P^*)$.

One can verify that $\tau_{(u,v)} \in O(P, \langle, \rangle)$ and $\tau_{(u,v)}^{-1}(p) = p + \langle u, p \rangle v - \langle v, p \rangle u$. The subgroup of $O(P, \langle, \rangle)$ generated by isotropic orthogonal transvections is denoted by $\text{Trans}_O(P)$.

Definition 3.12. The isotropic orthogonal transvections of $(P \perp R^2)$ of the form

$$\begin{aligned} (p, b, a) &\longmapsto (p - aq, b + \langle p, q \rangle, a), \\ (p, b, a) &\longmapsto (p - bq, b, a - \langle p, q \rangle), \end{aligned}$$

where $a, b \in R$ and $p, q \in P$, are called **elementary orthogonal transvections**.

The subgroup of $\text{Trans}_O(P \perp R^2)$ generated by the elementary orthogonal transvections is denoted by $E\text{Trans}_O(P \perp R^2)$.

3.14

Notation 3.13. In this paper P will denote either a finitely generated projective module of rank n , a symplectic module or an orthogonal module of even rank $n = 2m$ with a fixed form \langle, \rangle . And Q will denote $P \oplus R$ in the linear case and $P \perp R^2$ otherwise. We assume that $n \geq 2$, when dealing with linear case and symplectic case and $n \geq 4$ otherwise. We use notation $G(Q)$ to denote $\text{Aut}(Q)$, $\text{Sp}(Q, \langle, \rangle)$ respectively; $S(Q)$ will denote $SL(Q) = \{\sigma \in \text{Aut}(Q) : \wedge^n \sigma = 1\}$, $\text{Sp}(Q, \langle, \rangle)$ respectively; $T(Q)$ to denote $\text{Trans}(Q)$, $\text{Trans}_{Sp}(Q)$ respectively; and $ET(Q)$ to denote $E\text{Trans}(Q)$, $E\text{Trans}_{Sp}(Q)$ respectively.

3.15

Theorem 3.14. (Local-Global Principle for Transvection Groups) ([7, Theorem 3.6]) Let R be a commutative ring with identity and Q be as in Notation 3.13. Suppose $\sigma(X) \in G(Q[X])$ with $\sigma(0) = \text{Id}$. If for every maximal ideal \mathfrak{m} of R ,

$$\sigma_{\mathfrak{m}}(X) \in \begin{cases} E(n+1, R_{\mathfrak{m}}[X]) \text{ for linear case,} \\ E(n+2, R_{\mathfrak{m}}[X]) \text{ otherwise.} \end{cases}$$

Then $\sigma(X) \in ET(Q[X])$.

Theorem 3.15. ([7, Theorem 2]) $T(Q) = ET(Q)$. Hence $ET(Q)$ is normal subgroup of $G(Q)$.

3.16

Theorem 3.16. Let $\sigma \in S(Q)$ such that σ is homotopic to identity. Then $[\sigma, \tau] \in ET(Q)$ for all $\tau \in S(Q)$.

Proof : Since σ is homotopic to identity there exists $\varphi(X) \in S(Q[X])$ such that $\varphi(0) = \text{Id}$ and $\varphi(1) = \sigma$. Define

$$\Psi(X) = [\varphi(X), \tau]$$

Note that $\Psi(0) = Id$ and for every maximal ideal \mathfrak{m} of R , $\Psi(X)_{\mathfrak{m}} = [\varphi(X)_{\mathfrak{m}}, \tau_{\mathfrak{m}}]$. By Lemma 2.14, $\sigma_{\mathfrak{m}} \in E(n+1, R_{\mathfrak{m}})$ in linear case and $\sigma_{\mathfrak{m}} \in E(n+2, R_{\mathfrak{m}})$ in symplectic case. Since $E(n, R)$ is normal in $S(n, R)$, we have $\Psi(X)_{\mathfrak{m}} \in E(n+1, R_{\mathfrak{m}}[X])$ in linear case and $\Psi(X)_{\mathfrak{m}} \in E(n+2, R_{\mathfrak{m}}[X])$ in symplectic case. Thus by Theorem 3.14, $\Psi(X) \in ET(Q[X])$ which implies

$$\Psi(1) = [\varphi(1), \tau] = [\sigma, \tau] \in ET(Q).$$

□

Corollary 3.17. *Let R be a commutative ring and P be a finitely generated projective R -module of rank $n = 2m$. Then the group $\frac{S(Q[X], (X))}{ET(Q[X], (X))}$ is an abelian group.*

Proof : Let $\sigma(X), \tau(X) \in S(Q[X])$, we need to prove that $[\sigma(X), \tau(X)] \in ET(Q[X])$. Define,

$$\gamma(X) = [\sigma(X), \tau(X)]$$

Then $\gamma(0) = Id$. For every maximal ideal \mathfrak{m} of R , by Corollary 2.20, $[\sigma(X)_{\mathfrak{m}}, \tau(X)_{\mathfrak{m}}] \in E(n+1, R_{\mathfrak{m}}[X])$ in linear case and $[\sigma(X)_{\mathfrak{m}}, \tau(X)_{\mathfrak{m}}] \in E(n+2, R_{\mathfrak{m}}[X])$ in symplectic case. Hence by Theorem 3.14, $\gamma(X) \in ET(Q[X], (X))$.

□

3.18

Theorem 3.18. (Local-Global Principle for Transvection Groups in Relative case) ([3, Theorem 1.3]) *Let R be a commutative ring with identity and Q be as in Notation 3.13. Suppose $\sigma(X) \in G(Q[X], I[X])$ with $\sigma(0) = Id$. If for every maximal ideal \mathfrak{m} of R ,*

$$\sigma_{\mathfrak{m}}(X) \in \begin{cases} E(n+1, R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X]) & \text{for linear case,} \\ E(n+2, R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X]) & \text{otherwise.} \end{cases}$$

Then $\sigma(X) \in ETrans(Q[X], I[X])$.

3.19

Lemma 3.19. *Let R be a commutative ring and I be a proper ideal of R (i.e. $I \neq R$) and P be a finitely generated projective R -module of rank $n = 2m$. Then the group $\frac{S(Q[X], XI[X])}{ET(Q[X], XI[X])}$ is an abelian group.*

Proof : Let $\sigma(X), \tau(X) \in S(Q[X], XI[X])$, we need to prove that $[\sigma(X), \tau(X)] \in ET(Q[X], XI[X])$. Define,

$$\gamma(X) = [\sigma(X), \tau(X)]$$

Note that, $\gamma(0) = Id$. For every maximal ideal \mathfrak{m} of R , by Lemma 2.37, $[\sigma(X)_{\mathfrak{m}}, \tau(X)_{\mathfrak{m}}] \in E(n+1, R_{\mathfrak{m}}[X], XI_{\mathfrak{m}}[X])$ in linear case and $[\sigma(X)_{\mathfrak{m}}, \tau(X)_{\mathfrak{m}}] \in E(n+2, R_{\mathfrak{m}}[X], XI_{\mathfrak{m}}[X])$ in symplectic case. Thus $\gamma(X) \in ET(Q[X], XI[X])$ by Theorem 3.18.

□

4. Orthogonal groups

Throughout this section we will assume that $1/2 \in R$, where R is a commutative ring with identity.

Definition 4.1. Lower Central Series Let G be a group, and define $G_0 = G$, $G_n = [G_{n-1}, G]$ for $n \geq 1$. With these notations, we have

$$G = G_0 \supseteq G_1 \supseteq \cdots G_n \supseteq \cdots .$$

The above series of subgroups of G is called the lower central series of group G .

We say a group G is nilpotent if lower central series terminates after finitely many terms and if G_n is the first subgroup which is trivial in the series then G is said to be nilpotent of nilpotency class n .

Definition 4.2. Derived Series Let G be a group, and define $G^0 = G$, $G^n = [G^{n-1}, G^{n-1}]$ for $n \geq 1$. With these notations, we have

$$G = G^0 \supseteq G^1 \supseteq \cdots G^n \supseteq \cdots .$$

The above series of subgroups of G is called the derived series of group G .

We say a group G is solvable if derived series terminates after finitely many terms and if G^n is the first subgroup which is trivial in the series then G is said to be a solvable group of length n .

Definition 4.3. By $EO_R(Q \perp \mathbb{H}(P)) \cdot O_R(\mathbb{H}(P))$ we shall mean the subset $\{\sigma_1 \sigma_2 | \sigma_1 \in EO_R(Q \perp \mathbb{H}(P)), \sigma_2 \in O_R(\mathbb{H}(P))\}$ of $O_R(Q \perp \mathbb{H}(P))$.

4.3

Theorem 4.4. (Local-Global Principle for the Orthogonal groups) ([36, Theorem 4.2]) Let $m \geq 3$ and $\alpha(X) \in SO_{2m}(R[X])$, with $\alpha(0) = Id$. Then $\alpha(X) \in EO_{2m}(R[X])$ if and only if for any maximal ideal $\mathfrak{m} \subset R$, the canonical image of $\alpha(X)$ in $SO_{2m}(R_{\mathfrak{m}}[X])$ lies in $EO_{2m}(R_{\mathfrak{m}}[X])$.

4.4

Lemma 4.5. (R.A.Rao) ([27, Lemma 2.2]) Let R be a ring with Jacobson dimension $\leq d$. Let (Q, q) be a diagonalisable quadratic R -space. Consider the quadratic R -space $Q \perp \mathbb{H}(P)$, where $\text{rank } P > d$. Then,

$$O_R(Q \perp \mathbb{H}(P)) = EO_R(Q \perp \mathbb{H}(P)) \cdot O_R(\mathbb{H}(P)) = O_R(\mathbb{H}(P)) \cdot EO_R(Q \perp \mathbb{H}(P))$$

Definition 4.6. Spinor Norm Suppose R be a local ring and M be an R -module and B be a non-degenerate symmetric bilinear form. Let G be the orthogonal group corresponding to B . The spinor norm is a group homomorphism

$$SN : G \longrightarrow \frac{(R^*)}{(R^*)^2}.$$

The homomorphism is defined as follows: any element of G arising as reflection orthogonal to vector v is sent to the value $B(v, v)$ modulo $(R^*)^2$. This extends to a well-defined and unique homomorphism on all of G . The reflection orthogonal to vector v is defined as

$$\begin{aligned} \tau_v : M &\longrightarrow M \\ x &\mapsto x - 2v \frac{B(v, x)}{B(v, v)}, \text{ for all } v \in M. \end{aligned}$$

4.6

Observation 4.7. One can write the matrix $\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}$ as a product $\tau_{e_1 - e_2} \tau_{e_1 - ue_2}$. Hence its spinor norm is $4u$.

In view of ([22, Theorem 4]), $EO_{2m}(R)$ is a normal subgroup of $SO_{2m}(R)$ when R is a local ring.

4.7

Lemma 4.8. *Let R be a local ring and I be a proper ideal of R (i.e. $I \neq R$). If $\alpha \in SO_{2m}(R, I)$, then $\alpha^2 \in EO_{2m}(R, I)$.*

Proof : Let $\alpha \in SO_{2m}(R, I)$. Since $\alpha \in SO_{2m}(R, I)$, we can write $\alpha = Id + \alpha'$, for some $\alpha' \in M_{2m}(I)$. Let $\tilde{\alpha} = (Id, \alpha') \in SO_{2m}(R \oplus I, 0 \oplus I)$. By Lemma 2.26, $R \oplus I$ is a local ring. In view of Lemma 4.5 $SO_{2m}(R \oplus I) = SO_2(R \oplus I) \cdot EO_{2m}(R \oplus I)$. Since R is a commutative ring and by ([2, Lemma 3.2]) every element of $SO_2(R \oplus I)$ looks like

$$\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}, u \in (R \oplus I)^*.$$

Thus α^2 looks like

$$\begin{bmatrix} u^2 & 0 \\ 0 & u^{-2} \end{bmatrix}, u \in (R \oplus I)^*.$$

In view of observation 4.7, or otherwise, spinor norm of α^2 is $4u^2$, a square in $(R \oplus I)^*$. Thus by ([21, Theorem 6]) we have, $\alpha^2 \in EO_2(R \oplus I)$. (The details of the proof can be found in [22].)

$$\tilde{\alpha}^2 \in EO_{2m}(R \oplus I) \cap SO_{2m}(R \oplus I, 0 \oplus I) = EO_{2m}(R \oplus I, 0 \oplus I)$$

as $\frac{R \oplus I}{0 \oplus I} \simeq R$ is a retract of $R \oplus I$. Thus,

$$\tilde{\alpha}^2 = \prod_{k=1}^t b_k g e_{i_k j_k}(0, a_k) b_k^{-1}, b_k \in EO_{2m}(R \oplus I), a_k \in I.$$

Now, consider the homomorphism

$$f : R \oplus I \longrightarrow R$$

$$(r, i) \longmapsto r + i.$$

This f induces a map

$$\tilde{f} : EO_{2m}(R \oplus I, 0 \oplus I) \longrightarrow EO_{2m}(R, I).$$

Thus $\alpha^2 \in EO_{2m}(R, I)$. □

Corollary 4.9. *Let R be a local ring and I be a proper ideal of R (i.e. $I \neq R$). Then the group $\frac{SO_{2m}(R, I)}{EO_{2m}(R, I)}$ is an abelian group for all $m \geq 1$. In fact, every element of this group is of order 2.*

4.17

Lemma 4.10. *Let R be a local ring then the group $\frac{SO_{2m}(R)}{EO_{2m}(R)}$ is an abelian group for all $m \geq 1$. In fact, every element of this group is of order 2. In particular, $[SO_{2m}(R), SO_{2m}(R)] = EO_{2m}(R)$.*

Proof : In view of R.A. Rao's Lemma $SO_{2m}(R) = SO_2(R) \cdot EO_{2m}(R)$. Every element of $SO_2(R)$ looks like

$$\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}, u \in R^*.$$

Thus $SO_2(R)$ is an abelian group which implies that $\frac{SO_{2m}(R)}{EO_{2m}(R)}$ is also an abelian group. Since, for every element $\alpha \in SO_2(R)$, α^2 looks like

$$\begin{bmatrix} u^2 & 0 \\ 0 & u^{-2} \end{bmatrix}, u \in R^*.$$

Again spinor norm of α^2 is $4u^2$, a square in R^* . Thus by ([21, Theorem 6]) we have, $\alpha^2 \in EO_2(R)$. (The details of the proof can be found in [22].)

□

4.8

Theorem 4.11. *Let R be a local ring. Then the group $\frac{SO_{2m}(R[X])}{EO_{2m}(R[X])}$ is a solvable group of length at most 2.*

Proof : Let $\alpha(X), \beta(X) \in [SO_{2m}(R[X]), SO_{2m}(R[X])]$, we need to prove $[\alpha(X), \beta(X)] \in EO_{2m}(R[X])$. In view of Lemma 4.10, we may assume that $\alpha(0) = \beta(0) = Id$. Define,

$$\gamma(X, T) = [\alpha(XT), \beta(X)].$$

For every maximal ideal \mathfrak{m} of $R[X]$,

$$\gamma(X, T)_{\mathfrak{m}} = [\alpha(XT)_{\mathfrak{m}}, \beta(X)_{\mathfrak{m}}].$$

Since $\beta(X)_{\mathfrak{m}} \in [SO_{2m}(R[X]_{\mathfrak{m}}), SO_{2m}(R[X]_{\mathfrak{m}})] = EO_{2m}(R[X]_{\mathfrak{m}})$ and $EO_{2m}(R[X]_{\mathfrak{m}}) \trianglelefteq SO_{2m}(R[X]_{\mathfrak{m}})$, thus $\gamma(X, T)_{\mathfrak{m}} \in EO_{2m}(R[X]_{\mathfrak{m}}[T])$ and $\gamma(X, 0) = Id$. Thus by Theorem 4.4, $\gamma(X, T) \in EO_{2m}(R[X, T])$, by putting $T = 1$, one gets $\gamma(X, 1) = [\alpha(X), \beta(X)] \in EO_{2m}(R[X])$.

□

4.15

Theorem 4.12. *Let R be a local ring and I be a proper ideal of R (i.e. $I \neq R$). Then the group $\frac{SO_{2m}(R[X], I[X])}{EO_{2m}(R[X], I[X])}$ is a solvable group of length at most 2.*

Proof : Let $\alpha(X), \beta(X) \in [SO_{2m}(R[X], I[X]), SO_{2m}(R[X], I[X])]$, we need to prove that $[\alpha(X), \beta(X)] \in EO_{2m}(R[X], I[X])$. In view of Lemma 4.11, we may assume that $\alpha(0) = \beta(0) = Id$. Define,

$$\gamma(X, T) = [\alpha(XT), \beta(X)].$$

For every maximal ideal \mathfrak{m} of $R[X]$,

$$\gamma(X, T)_{\mathfrak{m}} = [\alpha(XT)_{\mathfrak{m}}, \beta(X)_{\mathfrak{m}}].$$

Since $\beta(X)_{\mathfrak{m}} \in [SO_{2m}(R[X]_{\mathfrak{m}}, I[X]_{\mathfrak{m}}), SO_{2m}(R[X]_{\mathfrak{m}}, I[X]_{\mathfrak{m}})] = EO_{2m}(R[X]_{\mathfrak{m}}, I[X]_{\mathfrak{m}})$. By ([36, Corollary 2.13]) $EO_{2m}(R[X]_{\mathfrak{m}}, I[X]_{\mathfrak{m}})$ is normal in $SO_{2m}(R[X]_{\mathfrak{m}}, I[X]_{\mathfrak{m}})$, and so $\gamma(X, T)_{\mathfrak{m}} \in EO_{2m}(R[X]_{\mathfrak{m}}[T], I[X]_{\mathfrak{m}}[T])$. Since $\gamma(X, 0) = Id$, by Theorem 2.36, $\gamma(X, T) \in EO_{2m}(R[X, T], I[X, T])$, by putting $T = 1$, one gets $\gamma(X, 1) = [\alpha(X), \beta(X)] \in EO_{2m}(R[X], I[X])$.

□

4.9

Theorem 4.13. *Let R be a commutative ring and P be a finitely generated projective R -module of rank $n = 2m$. Then the group $\frac{SO(Q[X], (X))}{ETrans_O(Q[X], (X))}$ is a solvable group of length at most 2, for $m \geq 2$, where $Q = P \perp R^2$.*

Proof : Let $\sigma(X), \tau(X) \in [SO(Q[X], (X)), SO(Q[X], (X))]$, we need to prove that $[\sigma(X), \tau(X)] \in ETrans_O(Q[X], (X))$. Define,

$$\gamma(X) = [\sigma(X), \tau(X)]$$

Then $\gamma(0) = Id$. For every maximal ideal \mathfrak{m} of R , $\sigma(X)_{\mathfrak{m}} \in [SO_{2m+2}(R_{\mathfrak{m}}[X]), SO_{2m+2}(R_{\mathfrak{m}}[X])]$, $\tau(X)_{\mathfrak{m}} \in [SO_{2m+2}(R_{\mathfrak{m}}[X]), SO_{2m+2}(R_{\mathfrak{m}}[X])]$. In view of Theorem 4.12, $\gamma(X)_{\mathfrak{m}} \in EO_{2m+2}(R_{\mathfrak{m}}[X], (X)_{\mathfrak{m}})$. Now using Theorem 3.14, we get $\gamma(X) \in ETrans_O(Q[X], (X))$. \square

Corollary 4.14. *Let R be a commutative ring and I be a proper ideal of R (i.e. $I \neq R$) and P be a finitely generated projective R -module of rank $n = 2m$. Then the group $\frac{SO(Q[X], XI[X])}{ETrans_O(Q[X], XI[X])}$ is a solvable group of length at most 2, for $m \geq 2$, where $Q = P \perp R^2$.*

Proof : Let $\sigma(X), \tau(X) \in [SO(Q[X], XI[X]), SO(Q[X], XI[X])]$, we need to prove that $[\sigma(X), \tau(X)] \in ETrans_O(Q[X], XI[X])$. Define,

$$\gamma(X) = [\sigma(X), \tau(X)]$$

Then $\gamma(0) = Id$. For every maximal ideal \mathfrak{m} of R , $\sigma(X)_{\mathfrak{m}} \in [SO_{2m+2}(R_{\mathfrak{m}}[X], XI_{\mathfrak{m}}[X]), SO_{2m+2}(R_{\mathfrak{m}}[X], XI_{\mathfrak{m}}[X])]$, $\tau(X)_{\mathfrak{m}} \in [SO_{2m+2}(R_{\mathfrak{m}}[X], XI_{\mathfrak{m}}[X]), SO_{2m+2}(R_{\mathfrak{m}}[X], XI_{\mathfrak{m}}[X])]$. In view of Theorem 4.12, $\gamma(X)_{\mathfrak{m}} \in EO_{2m+2}(R_{\mathfrak{m}}[X], XI_{\mathfrak{m}}[X])$. Now using Theorem 3.18 we get $\gamma(X) \in ETrans_O(Q[X], XI[X])$. \square

4.16

Lemma 4.15. *Let R be a local ring. If $\alpha(X), \beta(X) \in SO_{2m}(R[X])$ with $\alpha(0) = Id$, then $[\alpha(X), \beta(X)^2] \in EO_{2m}(R[X])$.*

Proof : Define,

$$\gamma(X, T) = [\alpha(XT), \beta(X)^2].$$

For every maximal ideal \mathfrak{m} of $R[X]$,

$$\gamma(X, T)_{\mathfrak{m}} = [\alpha(XT)_{\mathfrak{m}}, \beta(X)_{\mathfrak{m}}^2].$$

In view of ([21, Theorem 6]) $\beta(X)_{\mathfrak{m}}^2 \in EO_{2m}(R[X]_{\mathfrak{m}})$ and $EO_{2m}(R[X]_{\mathfrak{m}})$ is normal in $SO_{2m}(R[X]_{\mathfrak{m}})$, thus $\gamma(X, T)_{\mathfrak{m}} \in EO_{2m}(R[X]_{\mathfrak{m}}[T])$ and $\gamma(X, 0) = Id$. Thus by Theorem 4.4, $\gamma(X, T) \in EO_{2m}(R[X, T])$, by putting $T = 1$, one gets $\gamma(X, 1) = [\alpha(X), \beta(X)^2] \in EO_{2m}(R[X])$. \square

4.22

Corollary 4.16. *Let R be a local ring. Then the group $\frac{SO_{2m}(R[X], (X))}{EO_{2m}(R[X], (X))}$ is a nilpotent group of class at most 2.*

Proof : Let $\alpha(X) \in SO_{2m}(R[X], (X))$ and $\beta(X) \in [SO_{2m}(R[X], (X)), SO_{2m}(R[X], (X))]$, we need to prove that $[\alpha(X), \beta(X)] \in EO_{2m}(R[X], (X))$. Since in any group G , a commutator $[x, y] = xyx^{-1}y^{-1} = (xyx^{-1})^2x^2(x^{-1}y^{-1})^2$ is a product of squares. Therefore $\beta(X)$ can be written as a product of squares; and the result follows from Lemma 4.15.

We also give an alternative proof for this Corollary:

Let $\alpha(X) \in SO_{2m}(R[X], (X))$ and $\beta(X) \in [SO_{2m}(R[X], (X)), SO_{2m}(R[X], (X))]$, we need to prove that $[\alpha(X), \beta(X)] \in EO_{2m}(R[X], (X))$. Define,

$$\gamma(X, T) = [\alpha(XT), \beta(X)].$$

For every maximal ideal \mathfrak{m} of $R[X]$,

$$\gamma(X, T)_{\mathfrak{m}} = [\alpha(XT)_{\mathfrak{m}}, \beta(X)_{\mathfrak{m}}].$$

Since $\beta(X)_{\mathfrak{m}} \in [SO_{2m}(R[X]_{\mathfrak{m}}), SO_{2m}(R[X]_{\mathfrak{m}})] = EO_{2m}(R[X]_{\mathfrak{m}})$ and $EO_{2m}(R[X]_{\mathfrak{m}}) \trianglelefteq SO_{2m}(R[X]_{\mathfrak{m}})$, thus $\gamma(X, T)_{\mathfrak{m}} \in EO_{2m}(R[X]_{\mathfrak{m}}[T])$ and $\gamma(X, 0) = Id$ since $\alpha(0) = Id$. Thus by Theorem 4.4, $\gamma(X, T) \in EO_{2m}(R[X, T])$, by putting $T = 1$, one gets $\gamma(X, 1) = [\alpha(X), \beta(X)] \in EO_{2m}(R[X])$. Since $\gamma(0, 1) = Id$, thus $\gamma(X, 1) = [\alpha(X), \beta(X)] \in EO_{2m}(R[X], (X))$. □

We believe that the orthogonal quotients (in Theorem 4.11 and Theorem 4.13) are abelian groups, we show this when the base ring is a regular local ring containing a field.

Definition 4.17. Let k be a field. A ring R is said to be essentially of finite type over k if $R = S^{-1}C$, with S is a multiplicatively closed subset of C , and $C = k[x_1, \dots, x_m]/I$ is a quotient ring of a polynomial ring over k .

4.12

Proposition 4.18. Let R be a smooth affine algebra over a field k . If $\alpha(X) \in SO_{2m}(R[X])$ with $\alpha(0) = Id$, then $\alpha(X) \in EO_{2m}(R[X])$, for $m \geq 3$.

Proof : Let $\gamma(X, T) = \alpha(XT) \in SO_{2m}(R[X, T])$, then $\gamma(X, 0) = Id$. Thus by homotopy invariance (See [17], [18, Corollary 1.12], [32, Theorem 9.8].) we have $\gamma(X, 1) = \alpha(X) \in EO_{2m}(R[X])$. (The reader may also consult [13] for a version which is suitable for this application.) (There is a version for reductive groups in [33] which may be of independent interest.) □

The following Lemma is well known:

4.13

Lemma 4.19. Let R be an affine algebra over a field k . Suppose $R_{\mathfrak{p}}$ is regular local ring for some $\mathfrak{p} \in \text{Spec}(R)$. Then there exists $s \notin \mathfrak{p}$ such that R_s is a regular ring.

Proof : Let J be the Jacobian ideal of R . Then $V(J) = \text{Sing}(R)$. Since $R_{\mathfrak{p}}$ is regular local ring, $J \not\subseteq \mathfrak{p}$. Choose $s \in J \setminus \mathfrak{p}$. Now for every $\mathfrak{q} \in \text{Spec}(R)$ with $s \notin \mathfrak{q}$, we have $J \not\subseteq \mathfrak{q}$. Since every prime ideal of R_s looks like \mathfrak{q}_s , for some $\mathfrak{q} \in \text{Spec}(R)$ with $s \notin \mathfrak{q}$, we get $(R_s)_{\mathfrak{q}_s} = R_{\mathfrak{q}}$ is a regular local ring. Hence R_s is a regular ring. □

4.14

Theorem 4.20. Let R be a regular local ring essentially of finite type over a field k . If $\sigma(X) \in SO_{2m}(R[X])$, with $\sigma(0) = Id$, then $\sigma(X) \in EO_{2m}(R[X])$, for $m \geq 3$.

Proof : In view of Lemma 4.19, for every $\mathfrak{p} \in \text{Spec}(R)$ there exists $s \in R \setminus \mathfrak{p}$ such that R_s is a smooth algebra. Therefore, by Proposition 4.18, $\sigma_s(X) \in EO_{2m}(R_s[X])$, which implies $\sigma_{\mathfrak{p}}(X) \in EO_{2m}(R_{\mathfrak{p}}[X])$. Now, by Theorem 4.4 we have $\sigma(X) \in EO_{2m}(R[X])$. □

Corollary 4.21. *Let R be a geometric regular local ring containing a field. Then the group $\frac{SO_{2m}(R[X])}{EO_{2m}(R[X])}$ is an abelian group for $m \geq 3$.*

Proof : Let $\alpha(X), \beta(X) \in SO_{2m}(R[X])$, we need to prove that $[\alpha(X), \beta(X)] \in EO_{2m}(R[X])$. Define,

$$\gamma(X) = [\alpha(0), \beta(0)]^{-1}[\alpha(X), \beta(X)].$$

We will proceed by induction on $\dim R$. If $\dim R = 0$, then R is a field and the result follows from Proposition 4.18. Therefore we assume $\dim R \geq 1$. In [25], D. Popescu showed that if R is a geometric regular local ring then it is a filtered inductive limit of regular local rings essentially of finite type over a field.

Clearly, $\gamma(0) = Id$. Hence by Theorem 4.20, $\gamma(X) \in EO_{2m}(R[X])$. Now, using Lemma 4.10, we get $[\alpha(X), \beta(X)] \in EO_{2m}(R[X])$. □

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